Dynamics of Atom-Field Entanglement from Exact Solutions: Towards Strong-Coupling and Non-Markovian Regimes

Nick Cummings* and B. L. Hu†
Joint Quantum Institute and Department of Physics, University of Maryland, College Park, Maryland 20740
(Dated: August 16, 2007)

We examine the dynamics of bipartite entanglement between a two-level atom and the electromagnetic field. We treat the Jaynes-Cummings model with a single field mode and examine in detail the exact time evolution of entanglement, including cases where the atomic state is initially mixed and the atomic transition is detuned from resonance. We then explore the effects of other nearby modes by calculating the exact time evolution of entanglement in more complex systems with two, three, and five field modes. For these cases we can obtain exact solutions which include the strong-coupling regimes. Finally, we consider the entanglement of a two-level atom with the infinite collection of modes present in the intracavity field of a Fabre-Perot cavity. In contrast to the usual treatment of atom-field interactions with a continuum of modes using the Born-Markov approximation, our treatment in all cases describes the full non-Markovian dynamics of the atomic subsystem. Only when an analytic expression for the infinite mode case is desired do we need to make a weak coupling assumption which at long times approximates Markovian dynamics.

I. INTRODUCTION

In recent years the quest for a quantum computer has generated much interest in quantum information processing [1] and the need for a deeper understanding of the distinguishing quantum features of many body systems, such as quantum coherence and entanglement. Recent progress in many areas of physics, especially atomic-optical, condensed matter, low temperature and mesoscopic physics, has been brought about by new techniques (in manipulating atoms and light) and high precision measurements and fabrication of new materials. The drive for better preparation, manipulation and control on the experimental side is matched by an adjustment of focus and emphasis of theoretical studies. There is a trend to go back to simpler models (e.g., Bose-Hubbard for BEC) but explore the parameter regimes and physics hitherto unknown or unclear (such as the BCS-BEC transition, BEC-Mott transition) but made possible only recently by the high precision and highly manipulative experiments. With these efforts new demands on experimental designs (such as accurately positioning an atom in a cavity [2, 3]) for new devices (such as single atom laser [4, 5]) and new theoretical issues (such as quantum fluctuations and correlations, decoherence and disentanglement) come to the fore.

It is in this spirit that we explore some of these newly introduced issues in atom-field interaction, with the stated purpose of better understanding the effects of non-Markovian dynamics and strong coupling between a two-level atom (2LA) and the electromagnetic field (EMF). In this paper we will seek first to give a detailed, general, and exact treatment of the dynamics of entanglement in the Jaynes-Cummings model. From there, we expand our considerations to include the effects of other, nearby field modes, which can, in principle, add new features to the evolution of entanglement when coupling is strong enough. Finally, we will explore interaction between the atom and the full, infinite collection of modes in the intracavity field. In all these cases we solve for the exact dynamics [21] keeping the full unitary dynamics of the composite system, so that our treatment can capture the non-Markovian behavior of the atom (considered as a subsystem) even in the infinite-mode case. In this approach, then, we wish to give a detailed description of the much-studied, single-field-mode Jaynes-Cummings model and connect this to strong-coupling and non-Markovian phenomena in more general models. We want to adopt an approach which can best preserve the quantum coherence and entanglement of the system and include the full interplay of the subsystems involved (or back-action from its environment, if any one such subsystem merits special attention), treated self-consistently. The influence functional approach we take here is particularly suitable for such requirements, and they are necessary for a correct treatment of disentanglement, because in general it is an even more intricate and delicate issue than decoherence.

*Electronic address: nickc@umd.edu
†Electronic address: blhu@umd.edu

Simpler atom-field models, such as that of a 2LA interacting with one single EMF mode described by the Jaynes-Cummings model, have solutions in closed form. Such closed form solutions offer a good comparison with results involving approximations. Quantum entanglement shows up as 'collapse and revival' phenomena, and has been studied in depth by Pheonix and Knight [6, 7] and Gea-Banacloche [8] 15 years ago for initial pure states and Bose et al [9] for initial mixed states. We analyze this problem to obtain the exact time evolution of the entanglement (rather than just a bound on or average of entanglement) for general, mixed initial atomic states, and we explore the effect of detuning between the atomic transition frequency and the cavity mode frequency. This gives us a detailed and exact picture of the dynamics of entanglement in the well-known Jaynes-Cummings model.

Moving on to more complex models, we study the case of a 2LA interacting with two, three, or five EMF modes. We present an exact solution of the time evolution of entanglement showing the effect of these additional models for an arbitrary, pure initial atomic state. Using a perturbative approach we give leading order corrections to Jaynes-Cummings dynamics from other, nearby field modes in the case of strong atom-field coupling. There is a recent proposal to use the 2LA as a control qubit and the two EMF modes as a target qubit in the making of a CNOT gate [10]. Our calculation of entanglement between an atom with an initial mixed state and multiple field modes may be useful for these proposed studies.

In support of the new needs mentioned above, our work aims at probing the strong-coupling regime. Strong coupling is defined as $g >> \kappa, \gamma$, where κ is the cavity damping rate and γ is the rate of spontaneous emission from the atom. In this regime the coherent evolution of the atomic and intracavity field state is fast compared to the dissipation rates in the system, so coherent quantum effects become important [11]. This is the regime we explore in treating the few mode cases.

Beyond the strong-coupling regime, there is also the regime of super-strong coupling $g \simeq \Delta$, where Δ denotes free spectral range of the resonator in angular frequency. While, conventionally, strong coupling is achieved through recycling of the light by means of a high Q cavity, in the super strong regime the coupling between atoms and light is already strong during one round trip in the resonator. Thus the atom can affect the mode structure, in addition to the occupation of the modes, as is the case in conventional Cavity QED. (This, in our language is when the back-action of the atom on the field begins to become important.) According to [12], this super-strong-coupling regime can be achieved for a modest number of ultracold atoms in an optical lattice formed by standing waves in an optical resonator. Our studies of cases with a few field modes cover this super-strong regime for the case of a single atom, showing new behavior in this regime, and they apply to certain situations for the multi-atom scenario which is experimentally realistic.

Finally, we study the interaction of a 2LA with a cavity and its full, infinite collection of EMF modes in certain limits, going beyond a simple one- or few-mode treatment. We do not need to invoke the Born-Markov approximation and our approach generally applies to strong-coupling regimes. Our calculations yield formal solutions for the evolution of entanglement in terms of integrals, which admit numerical solutions in the general case, but in order to obtain simple analytic solutions we explore a solution for late-time behavior that is perturbative in the atom-field coupling, and, so, those expressions are limited to the weak-coupling regime of cavity QED and exhibit behavior corresponding to Markovian expectations.

The second aim for our work is to probe into the non-Markovian regimes. Since our system, comprised of one atom together with the modes of the EMF, is closed, the dynamics of the complete system is unitary. One may wonder where non-Markovian dynamics arises. This is a reasonable question because non-Markovian behavior is usually associated with open system dynamics, which is the case when one focuses on one of the subsystems, often the 2LA, and introduces some coarse-graining in the other subsystem, such as EMF, whence one begins to see dissipative and stochastic behavior in the open system. However, strictly speaking, non-Markovian just refers to history-dependent processes (involving memories), and it is not just for open systems, in the following sense: It is common knowledge in non-equilibrium statistical mechanics [13] that for two interacting subsystems the two ordinary differential equations governing each subsystem can be written as an integro-differential equation governing one such subsystem, thus rendering its dynamics non-Markovian, with the memory of the other subsystem's dynamics registered in the non-local kernels. All these are happening in a closed system except now one has shifted the attention to one of its subsystems. [22] When an exact solution for the combined system is available, as in many of the cases studied here, one can derive the non-Markovian dynamics in the corresponding cases when the above mentioned procedures are taken, or if the non-Markovian dynamics is available from an open system consideration, one can identify the coarse graining measures introduced. The commonly invoked Markovian approximation gives rise to dynamics farther remote from (meaning, less accurate than) the exact solutions.

We must add that investigations into the strong-coupling and non-Markovian regimes are two rather difficult tasks, even for cavity QED conditions. This study takes only a small step toward them. Our aim is to gain some intuitive understanding of the evolution of quantum entanglement in these new territories with a familiar and relatively solvable model. Much more work is needed to understand the general features for more complicated systems in these regimes.

In Sect. II we specify the details of the physical system we wish to treat and discuss how we will quantify the

entanglement present. Sect. III discusses the method we adapted to find the evolution of the system and the effects of initial conditions assumed. We then treat the specific cases of different collections of EMF modes in Sect. IV, including a field consisting of a single mode, a few modes, and the full collection of modes present in an idealized optical cavity. We end in Sec. V with a discussion and a summary.

II. SYSTEM AND QUANTITIES UNDER INVESTIGATION

A. Hamiltonian and Initial States Considered

The system we consider is a two-level atom (2LA) coupled to M electromagnetic field (EMF) modes. We adopt the dipole and rotating-wave approximations, neglecting motion of the atomic center of mass. The Hamiltonian of this system, as given by [14], is

$$\hat{H} = \hbar\omega_0 \hat{S}^+\hat{S}^- + \sum_{k=1}^{M} \hbar\omega_k \hat{b}^\dagger\hat{b} + \hbar(g_k\hat{S}^+\hat{b} + g_k^*\hat{S}^-\hat{b}^\dagger) \equiv \hat{H}_0 + \hat{H}_I \tag{1}$$

where

$$\hat{H}_0 = \hbar\omega_0 \hat{S}^+\hat{S}^- + \sum_{k=1}^{M} \hbar\omega_0 \hat{b}^\dagger\hat{b} \tag{2}$$

is the free Hamiltonian and

$$\hat{H}_I = \sum_{k=1}^{M} \hbar\delta_k \hat{b}^\dagger\hat{b} + \hbar(g_k\hat{S}^+\hat{b} + g_k^*\hat{S}^-\hat{b}^\dagger) \tag{3}$$

is the interaction Hamiltonian. Here ω_0 is the (bare) frequency of the atomic transition, ω_k is the frequency of the k^{th} field mode, g_k is the coupling of the k^{th} mode to the atom, and $\delta_k \equiv \omega_k - \omega_0$. Since $g_k \propto 1/\sqrt{\omega_k}$ implies $g_j = g_k\sqrt{\frac{\omega_k}{\omega_j}}$, we may choose to define things such that all values g_k are real. Where the atom and the electromagnetic field modes in question form a closed system, the dynamics of the total system is unitary. For the $M = 1$ case, this is just the familiar Jaynes-Cummings model with no dissipation.

We will assume a separable initial state described by the density matrix

$$\hat{\chi}(0) = \hat{\rho}_A(0) \otimes |0\rangle\langle 0| \tag{4}$$

where all electromagnetic field modes are in the vacuum state $|0\rangle$ and the atom may be in an arbitrary (possibly mixed) state, described by the atomic reduced density matrix $\hat{\rho}_A(0)$. If $\hat{\rho}_A(0)$ is pure then $\hat{\chi}(t)$ is pure for all times due to the unitary evolution.

Tracing over the field, the reduced density matrix of the 2LA at time t is given by $\hat{\rho}_A(t) \equiv Tr_F(\hat{\chi}(t))$. As is well known, it can be represented by a point in the Bloch sphere. We will use the spherical polar parametrization of the sphere, so that $\hat{\rho}_A$ can be specified by the triple (r, θ, ϕ), and, as usual, pure states lie on the surface of the sphere with $r = 1$. We choose $(1, 0, \phi)$ to represent the excited state of the atom $|e\rangle$ and $(1, \pi, \phi)$ to represent the ground state $|g\rangle$.

In this work we seek to quantify the bipartite entanglement between the atom and the electromagnetic field (considered as a whole). We will not look at entanglement with individual modes of the field separately, nor will we explore entanglement among the modes of the field. A local unitary operation of the form $\hat{U} = \hat{U}_{Atom} \otimes \hat{U}_{Field}$ does not change the amount of entanglement in the system [15], and the time evolution due to $\hat{H}_0$, which relates the Schrödinger picture to the interaction picture, is such a local unitary operation; therefore, we may work entirely in the interaction picture and compute the entanglement of the interaction picture state directly, as though it were the Schrödinger picture state.

B. Measures of Entanglement

When dealing with pure states, the Entropy of Entanglement (EOE) quantifies the amount of entanglement [15]. The EOE is defined to be the von Neumann entropy of the reduced density matrix of the system.

$$\mathcal{E}_{vN}(\hat{\chi}) = S(Tr_F[\hat{\chi}]) = S(\hat{\rho}_A) = -\sum_j p_j \log_2(p_j) \tag{5}$$

where the p_j represent the eigenvalues of the reduced density matrix $\hat{\rho}_A$. The EOE does not depend on which subsystem is traced over. We will always trace over the field degrees of freedom.

When the total state of the system is mixed, the EOE can no longer be used as an entanglement measure. While many measures of entanglement have been defined, including the Entanglement of Formation, the Distillable Entanglement, the Entanglement Cost, the I-Concurrence, and others, most of these involve an infimum or supremum over a large set (often the set of projector decompositions of $\hat{\chi}$) that is in general difficult or impossible to evaluate analytically and quite costly to compute numerically [15]. We choose to focus on an entanglement monotone known as the Logarithmic Negativity (LN) [16] because we can give explicit expressions for the value, and it can be easily calculated numerically [23].

The LN is defined as

$$\mathcal{E}_{\mathcal{N}}(\hat{\chi}) \equiv \log_2 \left\| \hat{\chi}^{T_B} \right\|_1 \tag{6}$$

where $\hat{\chi}^{T_B}$ is the partial transpose in the basis B of the density matrix and

$$\left\| \hat{O} \right\|_1 \equiv Tr\left(\sqrt{\hat{O}\hat{O}^\dagger} \right) \tag{7}$$

is the trace norm of the operator $\hat{O}$. The partial transpose in the basis $B = \{|j\rangle_A |k\rangle_F\}$ is defined such that if

$$\hat{\chi} = \sum_{j,k,l,m} \chi_{j,k,l,m} |j\rangle\langle l|_A \otimes |k\rangle\langle m|_F \tag{8}$$

then

$$\hat{\chi}^{T_B} = \sum_{j,k,l,m} \chi_{j,k,l,m} |j\rangle\langle l|_A \otimes |m\rangle\langle k|_F \tag{9}$$

so that the second subsystem has been transposed. While the operator produced by this procedure will depend on which subsystem is transposed and in which basis, the trace norm depends only on the eigenvalues, which will be independent of these choices [15].

In the case that the total state of the system is pure, the amount of entanglement can be determined from the reduced density matrix for one of the subsystems, as is clear from the definition of the EOE. In this case one may show, using the Schmidt decomposition, that the LN becomes

$$\mathcal{E}_{\mathcal{N}} = \log_2 \left(\left(\sum_j \sqrt{p_j} \right)^2 \right) \tag{10}$$

again with p_j as the eigenvalues of the reduced density matrix. When the total state of the system is mixed, however, knowing only the reduced density matrix is not sufficient (since, for example, the reduced density matrix of a maximally entangled state is the same as when the total system is in a separable, completely-mixed state). Thus, if we wish to track the evolution of entanglement for a total mixed state, we must keep more information than just the reduced density matrix.

III. ENTANGLEMENT EVOLUTION

A. Methods of Solution

For M=1 we have the Jaynes-Cummings model, and one can exactly diagonalize $\hat{H}_I$ in the basis known as the dressed states. The evolution of $\hat{\chi}$ in this basis is quite simple, and knowing $\hat{\chi}(t)$ we can then calculate the entanglement using the LN in general and the EOE if $\hat{\chi}(0)$ is a pure state. For more than one EMF mode, one can still diagonalize the Hamiltonian in the same way. Our initial state has all the electromagnetic modes in their vacuum, so the energy arises from the atomic state. This confines the initial state to a subspace spanned by the zero- and one-excitation energy eigenstates of $\hat{H}_0$. Because the total system evolves via a time-independent Hamiltonian and $[\hat{H}_0, \hat{H}_I] = 0$, we know that the system will remain in this subspace. We then only need to diagonalize $\hat{H}_I$ in the subspace of degenerate one-excitation eigenstates of $\hat{H}_0$. When there are M field modes, this subspace has dimension $M + 1$. While this diagonalization is straightforward numerically, doing this analytically requires the solution to a polynomial of degree

$M+1$, so for $M>3$ there is in general no closed form solution. In any case, having obtained the eigenvalues the solution is otherwise exact.

For a pure initial state $\hat{\chi}(0)$, however, we only need the reduced density matrix $\hat{\rho}_A$ to find the entanglement. Anastopoulos and Hu [14] found the solution for the evolution of $\hat{\rho}_A(t)$ in these types of systems with these initial conditions. Given an initial atomic state $(1,\theta,\phi)$ at time t the reduced density matrix of the atom interacting with M modes is given by

$$\hat{\rho}_A(t) = \begin{pmatrix} \frac{1}{2}u^*u(1+\cos(\theta)) & \frac{1}{2}ue^{i\phi}\sin(\theta) \\ \frac{1}{2}u^*e^{-i\phi}\sin(\theta) & 1-\frac{1}{2}u^*u(1+\cos(\theta)) \end{pmatrix} \tag{11}$$

where

$$u(t) = \mathcal{L}^{-1}\left(\frac{1}{z+i\omega_0+\widetilde{\mu}(z)}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{zt}}{z+i\omega_0+\widetilde{\mu}(z)}dz, \tag{12}$$

$\mathcal{L}^{-1}$ represents the inverse Laplace transform, and

$$\widetilde{\mu}(z) = \sum_{j=1}^{M}\frac{g_j^2}{z+i\omega_j}. \tag{13}$$

The entanglement between the atom and the field modes depends only on the eigenvalues of $\hat{\rho}_A(t)$,

$$p = \frac{1}{2}\left(1\pm\sqrt{1-4\cos^4\left(\frac{\theta}{4}\right)\left(|u|^2-|u|^4\right)}\right), \tag{14}$$

so it depends only on the norm of $u(t)$. For M field modes

$$|u(t)| = \left|\sum_{j=0}^{M}\frac{\prod_{k=1}^{M}(z_j+i\delta_k)}{\prod_{l\neq j}^{M}(z_j-z_l)}e^{z_jt}\right| \tag{15}$$

with the z_j values being all the solutions to the equation

$$z\prod_{k=1}^{M}(z+i\delta_k) + \sum_{k=1}^{M}\left(g_k^2\prod_{l\neq k}^{M}(z+i\delta_l)\right) = 0. \tag{16}$$

Using Eq. (14) we can compute the EOE or the LN. The LN takes the particularly simple form

$$\mathcal{E}_{\mathcal{N}} = \log_2\left(1+2\cos^2\left(\frac{\theta}{2}\right)\sqrt{|u|^2-|u|^4}\right) \tag{17}$$

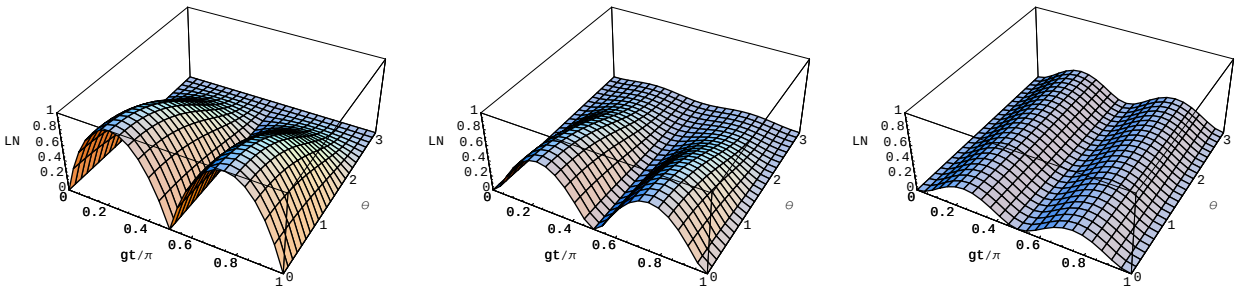
B. Dependence on Initial Conditions

In examining the evolution of entanglement in this system, one of the primary questions is how it depends on initial conditions. Given the Hamiltonian and the class of initial states we are considering, the entanglement at any point in time is completely independent of the parameter ϕ on the Bloch sphere for the initial state. While this is clear for pure states from the form of Eq. (14), we show this is true for mixed initial states as well. The unitary operator $\hat{U}_\phi(\alpha) \equiv e^{-i\hat{H}_0\alpha/(\hbar\omega_0)}$ acting on an initial state of the form Eq. (4) will simply shift the angle ϕ of the atomic state by an amount α. Consider two initial states, $\hat{\chi}(0)$ and $\hat{\chi}'(0) \equiv \hat{U}_\phi(\alpha)\hat{\chi}(0)\hat{U}_\phi(\alpha)^\dagger$, which differ only in their coordinate ϕ on the Bloch sphere. Since $\hat{H}_0$ commutes with $\hat{H}_I$, we know

$$\hat{\chi}'(t) = e^{-i\hat{H}_It/\hbar}\hat{U}_\phi(\alpha)\hat{\chi}(0)\hat{U}_\phi(\alpha)^\dagger e^{i\hat{H}_It/\hbar} = \hat{U}_\phi(\alpha)e^{-i\hat{H}_It/\hbar}\hat{\chi}(0)e^{i\hat{H}_It/\hbar}\hat{U}_\phi(\alpha)^\dagger = \hat{U}_\phi(\alpha)\hat{\chi}(t)\hat{U}_\phi(\alpha)^\dagger. \tag{18}$$

Because $\hat{U}_\phi(\alpha)$ represents a local operation, we also know that the degree of entanglement in the states $\hat{\chi}(t)$ and $\hat{\chi}'(t)$ must be the same.

For the case where the initial state of the system is a pure state, we can additionally conclude that, while the value of the entanglement at a given time depends on the θ coordinate of the initial state, qualitative features like the times at which local maxima and minima of entanglement occur will be the same for all such initial pure states. The easiest way to see this is by looking at Eq. (17), since the value of $|u(t)|$ at which the LN reaches extrema does not depend on θ.



(a) $r = 1$ (b) $r = 1/2$ (c) $r = 0$

Figure 1: LN as a function of time t and the initial state value of θ on the Bloch sphere for states with different values of r, where g is the value of the atom-field coupling. In the last case, of course, all values of θ correspond to the same state.

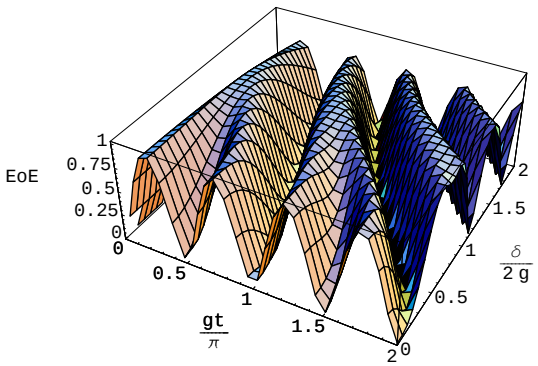


Figure 2: The dependence of entanglement on the parameters of the Hamiltonian, the detuning δ and atom-field coupling g. The odd numbered minima in the entanglement (as a function of time t) increase and eventually disappear as the detuning increases.

IV. SPECIFIC CASE STUDY

A. Single Mode — The Jaynes-Cummings Model

We begin with the case of a single mode of the field ($M = 1$) interacting with the atom in the familiar Jaynes-Cummings model. The Hamiltonian now depends only on the atom-field coupling g, the field mode frequency ω, and the atomic transition frequency $\omega + \delta$. First, consider a system in which the atom is initially in a pure, excited state and the field mode is resonant with the atomic transition. In this case, the system undergoes Rabi oscillations between the separable states $|e\rangle|0\rangle$ and $|g\rangle|1\rangle$, passing through a maximally entangled state each way. As a result, the entanglement has the relatively simple oscillatory behavior with a time scale set by g. For an initial pure state we have

$$u(t) = e^{-i(\omega+\frac{\delta}{2})t}\left[\left(\frac{\Delta\alpha+\delta}{2\Delta\alpha}\right)e^{-i\frac{\Delta\alpha}{2}t} + \left(\frac{\Delta\alpha-\delta}{2\Delta\alpha}\right)e^{i\frac{\Delta\alpha}{2}t}\right] \tag{19}$$

where $\Delta\alpha \equiv \sqrt{\delta^2 + 4g^2}$. For initially mixed atomic states, the fully density matrix for the bipartite system may be computed using the dressed states, as discussed in Sect. III A. The variation in the behavior of entanglement in the $\delta = 0$ case for different initial conditions is shown in Fig. 1. As stated above it does not depend on ϕ, and many of the qualitative features such as the times at which maximal and minimal entanglement occur do not depend on θ.

The other question is how the entanglement generated by the dynamics of the system depends on the parameters of the Hamiltonian. As is well known for the Jaynes-Cummings model, the time scale of the evolution is the vacuum effective Rabi frequency $\frac{1}{2}\sqrt{\delta^2 + 4g^2}$. What's more interesting is the dependence on the detuning δ shown in Fig. 2.

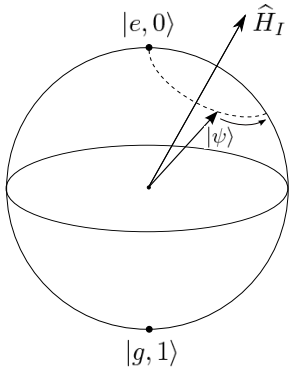


Figure 3: The Bloch sphere for the one-excitation manifold $\hat{H}_0$. The vector representing the quantum state of the system $|\psi\rangle$ precesses in a circle about the vector for the Hamiltonian $\hat{H}_I$ as time advances. When the state vector encounters either pole of the sphere, the state becomes separable. If the state vector crosses the equator of the sphere, the state is maximally entangled at that instant. The ratio of the detuning δ to the atom-field coupling g determines the angle the Hamiltonian vector makes with the vertical axis. If $\delta = 0$ the Hamiltonian vector is horizontal, and as δ/g becomes large the Hamiltonian vector approaches the vertical. The vector for the state starts on the pole of the sphere (due to the initial condition) and travels around a circle centered on the Hamiltonian vector; thus, the circle of precession is a great circle when $\delta = 0$, and the circle becomes very small when δ/g is large.

For an initial pure state, we can understand the time evolution of entanglement by focusing on the two level system composed of the one-excitation states of $\hat{H}_0$. This two level system can again be mapped to the Bloch sphere. As is well known, any time-independent Hamiltonian acting on this system can be represented by a fixed vector, and the state's evolution under the Hamiltonian can simply be described by a precession of the Bloch vector for the state about the Hamiltonian vector. If we choose the basis to be $B = \{|e\rangle|0\rangle, |g\rangle|1\rangle\}$ where the first state lies at the top of the Bloch sphere and the second state lies at the bottom, then we have

$$\hat{H}_I = \hbar \begin{pmatrix} -\delta/2 & g \\ g & \delta/2 \end{pmatrix} = -\frac{\delta}{2}\hat{\sigma}_z + g\hat{\sigma}_x = (g, 0, -\frac{\delta}{2}) \cdot \vec{\sigma}$$

where $\vec{\sigma}$ is the (pseudo) vector associated with the Pauli matrices, and $(g, 0, -\frac{\delta}{2})$ becomes the vector representing the Hamiltonian. Notice that all points along the "equator" of this Bloch sphere are maximally entangled, because they are all equivalent to $\frac{1}{2}(|e\rangle|0\rangle + |g\rangle|1\rangle)$ by a local unitary operation. When the detuning is small, the vector for $\hat{H}_I$ points almost along the x-axis, and the state rotates from $|e\rangle|0\rangle$ to $|g\rangle|1\rangle$ as it evolves in time, crossing the equator twice, so it passes through two maximally entangled states and two separable states during each rotation. Fig. 3 illustrates this Bloch sphere picture of the entanglement evolution, which accounts for all the qualitative features that appear in Fig. 2.

B. Multiple Modes

With additional modes of the electromagnetic field, the behavior of the system quickly becomes more complex. Let us begin with the bipartite entanglement between the atom and the EMF when the field has only two modes. First consider the "symmetric" case, where $g_1 = g_2 \equiv g$ and $\delta_1 = -\delta_2 \equiv \delta$. If the initial state is pure, then we have

$$u(t) = \frac{\delta^2 + 2g^2 \cos\left(t\sqrt{2g^2 + \delta^2}\right)}{\delta^2 + 2g^2}$$

and we can easily compute the entanglement as a function of time. If the initial atomic state is mixed, we can still compute the LN by the technique mentioned in III A. The dependence on initial conditions in Fig. 4 appears quite similar to the single mode case, while the dependence on δ shown in Fig. 4 is somewhat more complex. We are more interested, however, in cases where one of the field modes is very close to resonance, so a more practical case to investigate is that of three field modes.

In considering a finite number of field modes, we are primarily thinking of modeling the intracavity field in cavity QED, particularly in the optical regime. We will adopt the simplest possible model for such a cavity, two perfectly

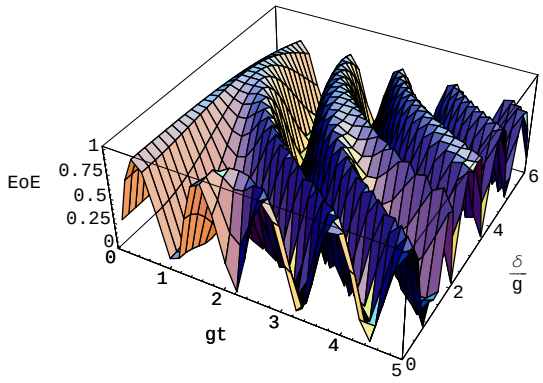


Figure 4: The dependence of the EOE on both time t and the detuning δ from the atomic resonance frequency for the case where there are two EMF modes.

conducting, infinite, parallel plates(the cavity model used in [14]). There are two classes of modes between the plates: the discrete set of modes with a purely longitudinal wave vector, which we will call the "cavity modes", and the continuum of modes with a transverse component, which propagate outward along the plates. If we ignore the coupling between the atom and these transverse traveling modes but rely on the fact that the effect of a single mode diminishes as it becomes detuned far from resonance, then we can try to approximate the cavity by only the finite number of modes M. This should reflect an actual cavity in the strong-coupling limit, where g is much larger than the spontaneous emission rate or the cavity decay rate.

Considering the frequencies of these cavity modes, if the mode closest to resonance is detuned by an amount δ then all other modes will be detuned by an amount $\delta + k\Delta$ where Δ is the free spectral range of the cavity in angular frequency, which is $\Delta = \frac{\pi c}{L}$ for our simple model. As noted earlier, all the g_j values are proportional to each other, so if we call the value for the mode closest to resonance g, this sets the value for all the other coupling constants. Thus, we are left with only four constants whose values need be specified.

We will consider the regime where $\delta \ll \Delta$ (as is often the case in cavity QED experiments); thus, one mode is almost in resonance, and the next nearest two modes have almost the same detuning. As a result, it will make sense to consider only odd total numbers of modes M in trying to understand the cavity. With this further specificity, we may rewrite the formula for $|u(t)|$. Let $M = 2Q + 1$, then

$$|u(t)| = \left| \sum_{j=0}^{M} \frac{\prod_{n=1}^{Q} \left(w_j^2 + n^2\Delta^2\right)}{\prod_{l\neq j}^{M} \left(w_j - w_l\right)} e^{w_j t} \right| \tag{20}$$

where w_j are solutions to the equation

$$\left(w^2 - i\delta w + g^2\right) \prod_{n=1}^{Q} \left(w^2 + n^2\Delta^2\right) + 2wg^2 \sum_{n=1}^{Q} \left(1 - \frac{n^2\Delta^2}{\left(\omega_0 + \delta\right)^2}\right)^{-1} \left(w + \frac{in^2\Delta^2}{\left(\omega_0 + \delta\right)}\right) \prod_{j\neq n}^{Q} \left(w^2 + j^2\Delta^2\right) = 0. \tag{21}$$

We consider first the $M = 3$ case. When Δ is small, the time evolution behavior, shown in Fig. 5(a), becomes considerably more complex with the contributions of more frequency components to $u(t)$. The effect of the addition of two more modes in the $M = 5$ case is shown in Fig. 5(b). The clear intuitive interpretation that was present in the single mode case is not obvious in these cases.

In optical frequency cavity QED experiments, one is generally working in the regime where $\omega_0 \gg \Delta \gg g$ and δ is smaller than or on the same order as g. In this case, other cavity modes are far enough away in frequency to have little effect on the atom, and the evolution for the multi-mode models becomes nearly identical to that of the single mode case. We can calculate the poles in this regime by writing Eq. (21) in terms of the dimensionless parameters g/Δ, Δ/ω_0, and δ/ω_0 and solving it perturbatively in these small parameters. In experiments, these parameters might

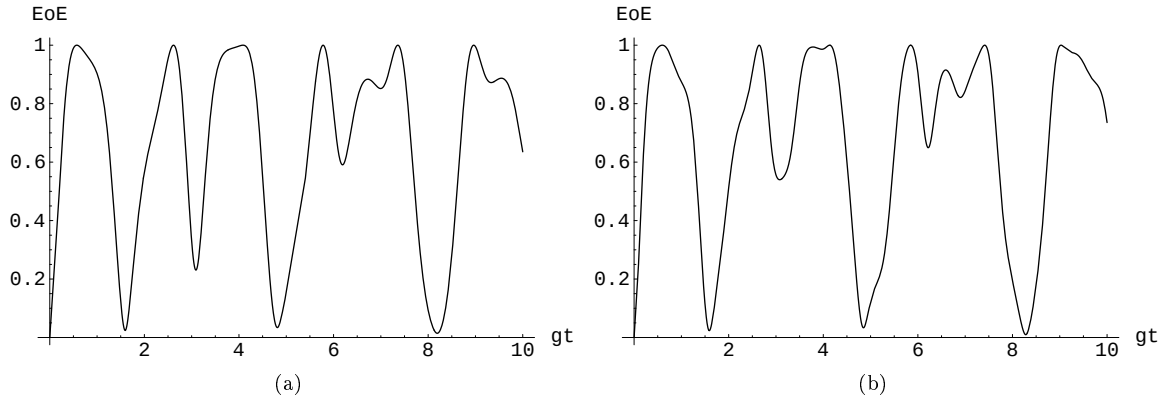


Figure 5: The evolution with time t of the EOE between an atom in an initially pure, excited state and (a) three modes or (b) five modes of the EMF. To highlight the effect of additional modes, we choose the values for the detuning δ, atom-field coupling g, and free spectral range Δ so that $\Delta/g = 5$, $g/\delta = 10$, and $\omega_0/g = 10^7$, though these are unlikely to be characteristic of experimental values.

typically have values on the order of 10^{-4}, 10^{-3}, and 10^{-7} respectively. This perturbative analysis leads to the poles

$$w = -\frac{i}{2}\left(\delta \pm \sqrt{\delta^2 + 4g^2}\right) \tag{22}$$

$$w = \pm i\left(k\Delta + \frac{g^2}{k\Delta}\right) \qquad k \in \left\{1, 2, ..., \frac{M-1}{2}\right\} \tag{23}$$

keeping up to terms linear in δ/ω_0 and terms quadratic in g/Δ and Δ/ω_0 (including terms of order g/ω_0). This is the lowest order at which the poles in Eq. (23) yield non-vanishing contributions to $|u(t)|$ [24]. In the sum from Eq. (20), the poles of the form shown in Eq. (22) give terms of order unity, while poles of the form shown in Eq. (23) yield terms of order g^2/Δ^2, so in this regime the effects of additional modes are small, but we have found an analytic expression for the behavior of the system for an arbitrary, finite number of modes.

C. Full Intracavity Field

Another approach toward understanding the effect of the cavity is to look at the long-time behavior of entanglement in the presence of the infinite field modes of the cavity. Again, we will use the simple cavity model mentioned earlier consisting of two infinitely large, perfectly conducting parallel plates. We can begin with a box having a square transverse cross-section of area A and a longitudinal length L with boundary conditions such that the field vanishes on the plates at the longitudinal boundaries and is periodic on the transverse boundaries. In this case we can write $g_{\vec{k}} = \frac{\lambda}{\sqrt{\omega_{\vec{k}} LA}}$, where the dimensionless quantity λ is the strength of the coupling to the overall field. As found in [14],

$$\mu(s) = \sum_{\vec{k}} g_{\vec{k}}^2 e^{-i\omega_{\vec{k}} s} \to \frac{\lambda^2}{2\pi^2 L} \sum_{n=-\infty}^{\infty} \int_{|\frac{\pi n}{L}|}^{\infty} e^{-ik's} dk'$$

in the continuum limit where $A \to \infty$. Since this integral clearly diverges, we add an exponential cutoff by taking $s \to s - i\epsilon$ to regularize it. With this cutoff

$$\mu_\epsilon(s) = \frac{\lambda^2}{2\pi^2 L} \sum_{n=-\infty}^{\infty} \int_{|\frac{\pi n}{L}|}^{\infty} e^{-ik's - k'\epsilon} dk' = \frac{\lambda^2}{\pi L} \frac{1}{\epsilon + is} \frac{1 + e^{-i\pi(s-i\epsilon)/L}}{1 - e^{-i\pi(s-i\epsilon)/L}}. \tag{24}$$

Taking the Laplace transform yields approximately [14]

$$\widetilde{\mu_\epsilon}(z) = \frac{-i\lambda^2}{\pi^2 \epsilon} + \frac{\lambda^2}{\pi^2} z \ln\left(ie^{\gamma_e} \epsilon z\right) - \frac{i\lambda^2}{\pi L}\left[\ln\left(\Gamma\left(\frac{Lz}{i\pi}\right)\right) - \frac{Lz}{i\pi}\ln\left(\frac{Lz}{i\pi}\right) + \frac{Lz}{i\pi} + \frac{1}{2}\ln\left(\frac{Lz}{2i\pi^2}\right)\right] + \mathcal{O}(\epsilon) \tag{25}$$

where $\ln(\Gamma(z))$ is defined such that it has a branch cut on each ray $\{-n + iy \,|\, n \in \mathbb{N} \,\&\, y \in \mathbb{R} \geq 0\}$, and γ_e is the Euler-Mascheroni constant. Clearly this expression depends on the cutoff ϵ, which we assume to be small. One cannot

in the end take the limit of $\epsilon \to 0$ for this model, even with renormalization of the model parameters. ϵ should be regarded as a phenomenological parameter that reflects the fact that at high frequencies the approximations that underlie our model (the rotating wave approximation, the two level approximation, etc.) must give way to other physics.

Inserting this result for $\widetilde{\mu}(z)$ into Eq. (12), $u(t)$ may be obtained by finding the residues arising from the poles of the integrand. As found in [14], all poles have a negative real part. From Eq. (15) one can see that the negative real part of each pole will cause its contribution to $u(t)$ to die off exponentially with time, so the pole with the greatest real part will dominate at long times.

If λ is sufficiently small, this pole can be found by doing a perturbative expansion in λ that yields

$$z_p = -i\widetilde{\omega}_0 - \widetilde{\mu}_\epsilon(-i\widetilde{\omega}_0) + \mathcal{O}\left(\lambda^4\right) = -i\Omega - \gamma \tag{26}$$

where $\widetilde{\omega}_0 \equiv \omega_0 - \frac{\lambda^2}{\pi^2\epsilon}$ for notational convenience. Let us further define

$$\Omega_\infty \equiv \widetilde{\omega}_0 - \frac{\lambda^2\widetilde{\omega}_0}{\pi^2}\ln\left(e^{\gamma_e}\epsilon\widetilde{\omega}_0\right). \tag{27}$$

Ω_∞ would be the value of Ω in the case of the free field and, therefore, the physically observable dressed value of the two-level transition frequency. If we rewrite our expression for z_p in terms of this dressed atomic frequency, we may write

$$\widetilde{\omega}_0 = \Omega_\infty + \frac{\lambda^2\Omega_\infty}{\pi^2}\ln\left(e^{\gamma_e}\epsilon\Omega_\infty\right) + \mathcal{O}\left(\lambda^4\right), \tag{28}$$

so then the pole from Eq. (26) becomes

$$z_p = -i\Omega_\infty - \frac{i\lambda^2}{\pi L}\left\{\ln\left[\Gamma\left(-\frac{L\Omega_\infty}{\pi}\right)\right] + \frac{L\Omega_\infty}{\pi}\ln\left(-\frac{L\Omega_\infty}{\pi}\right) - \frac{L\Omega_\infty}{\pi} + \frac{1}{2}\ln\left(-\frac{L\Omega_\infty}{2\pi^2}\right)\right\} + \mathcal{O}\left(\epsilon\right) + \mathcal{O}\left(\lambda^4\right). \tag{29}$$

This perturbative solution will be valid as long as λ is sufficiently small that the higher order terms can be ignored and no branch cut lies between $-i\Omega_\infty$ and the pole calculated in Eq. (29). Clearly, the expression will not be valid when $L\Omega_\infty/\pi$ is a non-negative integer, which is the condition for resonance. When Ω_∞ is sufficiently close to resonance, we can verify numerically that there will be two closely spaced poles, as one would expect from a simple Jaynes-Cummings model treatment.

In the case that the perturbative solution is valid and gives the only pole important at long times, then in that limit $|u(t)| \propto e^{-\gamma t}$ from Eq. (26). The value of γ depends on the value of the quantities Ω_∞, λ, ϵ, and L. When λ is sufficiently small, γ may be obtained by taking the real part of Eq. (29). In other cases it may be found numerically. Having determined the poles, if they are first order then the residue for each will simply be $\left(1 + \frac{d\widetilde{\mu}(z)}{dz}\big|_{z_p}\right)^{-1}$. Combining the expression for $|u(t)|$ at long times with our expressions for entanglement of a pure state yields a relatively simple behavior for the entanglement. The exponential decay of atomic coherence noted by [14] leads to entanglement that falls off toward zero at long times.

V. SUMMARY AND DISCUSSION

In this work we have considered the interaction between a two-level atom and different numbers of electromagnetic field modes with the aim of gaining a detailed description and deeper understanding of the nature and dynamics of quantum entanglement between the atom and the field. We have produced exact results (given the usual atom-field interaction Hamiltonian derived from making the rotating wave, dipole, and two-level approximations) that go beyond the usual single-mode Jaynes-Cummings model and give the complete development of quantum entanglement in time for a system where the field is initially in the vacuum state.

On the effect of initial conditions of the atom in the general case, we find that quantum entanglement is not affected by the ϕ angle of the initial atomic state on the Bloch sphere. For initially pure atomic states, we also find that qualitative features of the time evolution of entanglement remain the same for different values of θ.

For the case where the cavity has only one dominant field mode, our calculations reproduce the familiar results for the Jaynes-Cummings model obtained before [6, 8, 17]. Our result shows the expected oscillations of entanglement between the atom and the field, including periodic complete disentanglement in the resonant case. While many prior treatments assume a pure initial atom-field state or that the field mode is resonant with the 2LA, we have calculated the entanglement when the initial state (and, thus, the state at all times) of the atom-field system is mixed. We have

also shown the effects of detuning from resonance on the dynamics, giving a simple conceptual picture that accounts for all qualitative features.

For the situation when the atom is coupled to a finite number of electromagnetic field modes, since our calculation made no weak-coupling approximation, we are able to give a detailed account of the behavior of entanglement in the strong-coupling regime. Indeed, if one wishes to relate this to an actual cavity QED system where losses due to spontaneous emission and cavity damping are present, our closed-system treatment will be useful exclusively in the case of strong coupling $g \gg \kappa, \gamma$ at sufficiently early times where $\kappa t \ll 1$ and $\gamma t \ll 1$, wherein dissipation is insignificant and can be ignored.

We have discussed how the effects of these additional modes far from resonance are small in the normal parameter range of current cavity QED experiments. On the other hand, if the system were placed in the super-strong-coupling regime, where g is comparable to the free spectral range of the cavity Δ, then we can see from Fig. 5 that the dynamics are significantly altered from those of the single-mode case. Our calculations add new, significant features to the behavior of the entanglement.

It has been suggested [12] that one may be able to reach the super-strong-coupling regime experimentally in a system with many ultracold atoms trapped in an optical lattice inside the cavity. Under the right circumstances, our calculations might apply to such a system. If the experimental realization were such that the intracavity field coupled almost identically to the internal states of all N atoms so that the system behaved like the Tavis-Cummings model [18], then our model could still be applied if the atoms were initially prepared in the Dicke state $|N/2, -(N-1)/2\rangle$ [19] (i.e., a collective state with one excitation symmetrized over all atoms). In this case, the collective dipole of the atomic sample behaves like a single, two-level atom with enhanced coupling constant $g \to g\sqrt{N}$. Of course, in general the situation may be considerably more complex than a Tavis-Cummings type model, but at the least our results suggest interesting new dynamics for quantum entanglement in this super-strong-coupling regime.

Finally, we have treated the case of a 2LA interacting with the infinite number of modes present in an intracavity field and given an account of the long-time behavior of their entanglement. In this model we have assumed that the mirrors are perfectly reflecting. There are two possible approaches to applying these results to a physical cavity with damping. If the damping time scale is sufficiently longer than the other important dynamical time scales then the damping can be ignored for times $\kappa t \ll 1$, and one can apply our results directly. Alternately, one could modify the treatment we have given by using cavity field modes that extend through the cavity mirrors to the exterior, as would exist with partially transmitting mirrors, though the entanglement calculated would still be with the entire field, not merely the intracavity portion.

Though our solution for the cavity case could be evaluated numerically at any time, we pay special attention to the long-time behavior where an analytic result can be found. In order for this regime to be meaningfully applied to a physical system with $\kappa \neq 0$, one requires that such long times are still significantly shorter than $1/\kappa$. The long-time solution we have presented requires all other contributions to $u(t)$ to be negligibly small. These correspond to poles in the complex plane with negative real parts that are larger in absolute value. If the next most significant term has a real part $-\gamma'$, then the time scale that defines long times will be $(\gamma' - \gamma)^{-1}$. Given that we expect any other poles to move off to infinity in the complex plane as $\lambda \to 0$, if λ is small enough the time scale that defines long times should become comparatively short, and there should be a window of validity where it can be considered long times and yet κ may be ignored.

In obtaining these exact solutions, we are able to see several features of the dynamics. With the infinite collection of modes, we find that the terms from Eq. (12) are exponentially decaying, leading to a behavior in which the entanglement dissipates at long times. By contrast, when only a few field modes are included each term is purely oscillatory and no such decay is present. In each of the cases with an infinite collection of field modes, our exact calculation leads to the expression for $\widetilde{\mu}(z)$ that can be used to evaluate $u(t)$ numerically in order to obtain the behavior of the system. To give more concrete results, we have examined the long-time behavior working out the result perturbatively in the coupling λ. While this long-time behavior results in an exponential decay of coherence in the atom [14] characteristic of Markovian behavior of the atom (now considered as an open system with the field modes integrated away), our exact result for the total system can provide a check on how good the various approximations introduced for any description of the open system could be. These features could be explored by solving the equations we derived here numerically in particular limits of interest.

We addressed the strong-coupling regime above. Another important feature which our study of the full dynamics of the total system can provide some comparison with is the non-Markovian versus Markovian dynamics of a related open system. As is explained in the Introduction a la Zwanzig's projection operator paradigm, the exact and complete description of the total system (via an ordinary differential equation) can be transcribed to that of a distinguished subsystem (where one desires a detailed description) with the dynamics of all the other subsystems it interacts with subsumed in the non local kernels of integro-differential equation it obeys. This is the origin of non-Markovian dynamics with memory in a closed system. The open-system dynamics is only one step away from this, when one decides to coarse-grain away to varying extents the degrees of freedom of the other subsystems the distinguished

system interacts with. That is why the dynamics of such open systems are generically non-Markovian dynamics. The unitary dynamics of a closed combined system thus offers the best way to compare the relation of the non-Markovian dynamics of the open system with the Markovian dynamics of the same open system and assess the accuracy of various approximations introduced for its description. In the cases that include a finite number of field modes, one can obtain such subsystem dynamics quite directly from our solution for the total system dynamics. For a long time quantum atomic-optical experiments have operated in parameter ranges where the dynamics give relatively simple yet highly accurate results, so these issues have been largely ignored. But with stronger couplings, shorter times, and many correlated atoms present (the case of cold atoms in optical lattices is a familiar one) this luxury may not be available for too long.

The ever increasing capability for exquisite control of atom-field interactions in the laboratory and the increasing demands of quantum information applications require ever increasing accuracy in our theoretical modeling of these quantum systems. In addition, optical cavity QED is now truly approaching the regime of strong coupling. By working out the dynamics of quantum entanglement in this simple system exactly, in the regime of strong coupling and without making any compromising approximations (like the popular Born-Markov approximation), we hope to begin to uncover some new features of this realm of physics. Ours is really just a small step in this direction. There remains much work to be done, both to understand the generic behaviors of these simple systems by model studies and to add more features to the theoretical models that can provide a closer depiction of reality, captured in the near future by higher precision experiments.

Acknowledgments

We are indebted to Charis Anastopoulos for very detailed and helpful discussions. NC thanks Luis Orozco and Perry Rice for useful comments. This work is supported in part by grants from NSA-LPS, NIST and NSF-ITR program (PHY-0426696).

---

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 2000).
[2] B. W. S. Nußmann, M. Hijlkema, Phys. Rev. Lett. **95**, 173602 (2005).
[3] H. J. K. T. W. Lynn, K. Birnbaum, J. Opt. B Quant. Semiclass. Opt , quant-ph/0507064.
[4] J. Mckeever et al., Nature **425**, 268 (2003).
[5] B. Sun, M. S. Chapman, and L. You, Phys. Rev. A **69**, 042316 (2004).
[6] S. J. D. Phoenix and P. L. Knight, Phys. Rev. A **44**, 6023 (1991), Phys. Rev. Lett. 66, 2833 (1991).
[7] P. V. Buzek, H. Moya-Cessa, Phys. Rev. A **45**, 8190 (1992).
[8] J. Gea-Banacloche, Phys. Rev. Lett. **65**, 3385 (1990).
[9] S. Bose et al., Phys. Rev. Lett. **87**, 050401 (2001), Erratum: 87, 279901 (2001).
[10] F. Saif et al., quant-ph/0605178.
[11] P. Maunz et al., Phys. Rev. Lett. **94**, 033002 (2005).
[12] D. Meiser and P. Meystre, quant-ph/0605020.
[13] R. Zwanzig, *Nonequilibrium Statistical Mechanics* (Oxford University Press, 2001).
[14] C. Anastopoulos and B. L. Hu, Phys. Rev. A **62**, 033821 (2000), quant-ph/9901078.
[15] M. B. Plenio and S. Virmani, (2006), quant-ph/0504163.
[16] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[17] J. Gea-Banacloche, Phys. Rev. A **47**, 2221 (1993), 44, 5913 (1991).
[18] M. Tavis and F. W. Cummings, Phys. Rev. **170**, 379 (1968).
[19] R. H. Dicke, Phys. Rev. **93**, 99 (1954).
[20] M. Horodecki, P. Horodecki, and R. Horodecki, *Quantum Information: An Introduction to Basic Theoretical Concepts and Experiments* (Springer, 2001), chap. 5, quant-ph/0109124.
[21] Exact up to the approximations inherent in the usual model for the Hamiltonian of the system, which we will discuss
[22] Should the other subsystem possess a much greater number of degrees of freedom (called environment) and are coarse-grained in some way, the non local kernels are then responsible for the appearance of dissipation and noise. It is the act of coarse-graining which turns this particular subsystem of interest open.
[23] However, the LN only quantifies the degree to which the state in question violates the Peres-Horodecki positive partial transpose condition, and it is known that for bipartite quantum systems with a total Hilbert space dimension greater than six there exist entangled states, known as bound entangled states, that have positive partial transpose and, therefore, a LN of zero [20].
[24] If one keeps only to zeroth order in δ/ω_0 and linear order in g/Δ and Δ/ω_0, then the resulting expression for $|u(t)|$ is identical to the single-mode case, as one may expect when all other modes are very far off resonance.